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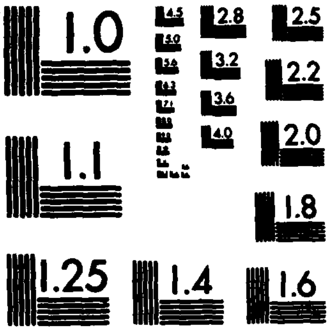

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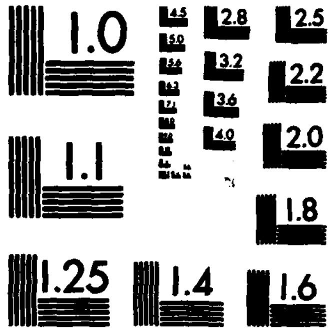

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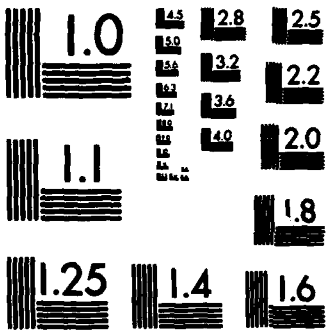

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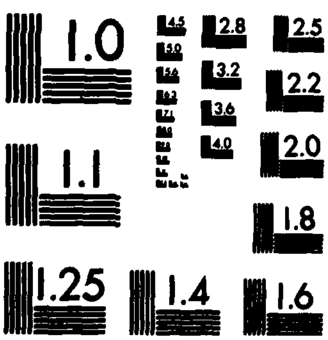

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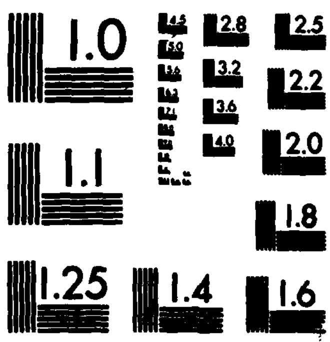

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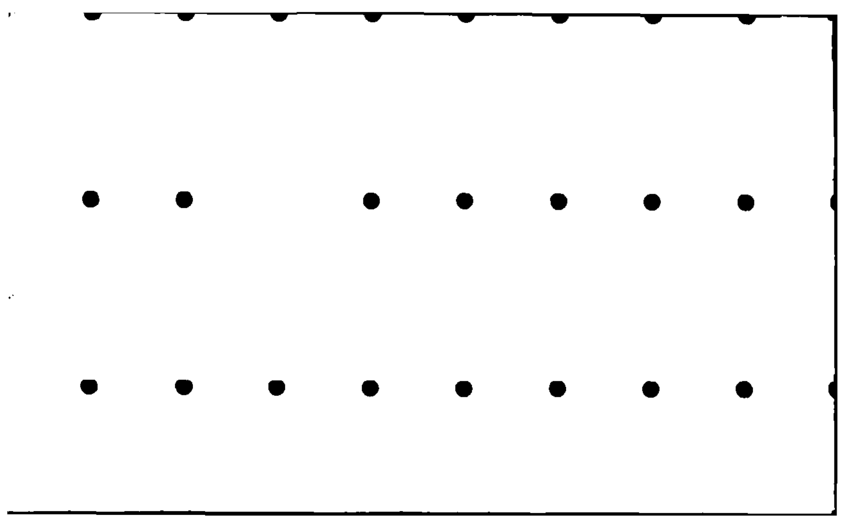

THE CONTROLLABILITY OF BILINEAR SYSTEMS.

PART I: COMPLETE CONTROLLABILITY OF HOMOGENEOUS SYSTEMS IN $\mathbb{R}^2-\{0\}$.

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1. Introduction.

This paper is written in the spirit of obtaining a concrete characterization of a system about which a fair bit in general, but not much in particular is known. The last decade has witnessed a growing propensity among systems theorists for the language and techniques of differential geometry, particularly applied to problems involving the structure of nonlinear control systems. Initially, this research addressed itself to the task of classification, and produced a literature concerned with such properties as minimality and realizability. In contrast, it is fair to say that very little work concerning the qualitative behavior of these systems had been attempted until recently. Since some early papers in the sixties [17, 20], it is only in the last few years that concepts like controllability [2, 6, 10, 12, 14, 13] or stabilizability [7, 11, 19, 21] have begun to receive attention in the literature.

Beyond a certain level of abstraction it is probably impossible to proceed in the investigation of nonlinear systems without adopting the perspective of differential geometry as is now customary. But such work must rely upon insights developed in very concrete settings to be of real use. Here, we report on an investigation at that opposite pole of abstraction. Specifically, we consider the second order homogeneous bilinear system

$$\dot{x} = Ax + uDx \triangleq h(x,u) \tag{1}$$

where $x \in \mathbb{R}^2$, A and D $\in \mathbb{R}^{2 \times 2}$, and u is a piecewise continuous function taking values in $\mathbb{R}$. The principal contribution is a statement of necessary and sufficient conditions for complete controllability of this system, given by Theorem 1 in section 3.

As well as holding significance for the more general nonlinear setting,

equations of this kind have a well reported range of applications in their own right [5, 18]. The authors' interest in such systems was sparked when earlier work on the the stability of second order quadratic differential equations [16] necessitated a consideration of bilinear-type structure quite independently. That result afforded a complete characterization of conditions for the stabilizability of planar bilinear systems under constant linear state feedback [15].

~~The present work~~ This report may be seen as a part of ~~our~~ a continuing effort to address some typical problems of nonlinear control, initially in settings simple enough to permit complete solutions based upon algebraic statements involving system parameters. It is ~~our strong feeling~~ felt that much more effort of this nature will be required before the fruit of theoretical work in nonlinear control theory becomes useful or even accessible for application in practical situations.

This report is organized as follows. A preliminary discussion of techniques and terminology is given in section 2. The question of complete controllability is resolved in section 3. In section 4 we consider a variety of related controllability concepts. Future reports will concern the nature of reachable sets when controllability fails, as well as similar results for general bilinear systems on the plane.

2. Definitions and Preliminary Discussion.

All of the results to follow could be cast in a more abstract form by an appeal to the theory of vector fields on manifolds. However, the chief contribution of this paper, as described in the introduction, consists in the classification of planar bilinear systems according to concrete algebraic conditions derived from the simplest notions of linear algebra and matrix

theory. We will indicate the relation of these conditions to the more abstract literature where appropriate. But, to our knowledge, the concepts central (although almost certainly not peculiar) to the controllability of system (1) as developed in the sequel have not yet been explored in the differential geometric control literature.

Even in our simple setting, discussion is greatly facilitated by a number of terminological conventions, which it is hoped will not distract the reader's attention from the underlying elementary geometric motivation. In this section we will introduce that terminology and establish some useful basic properties encouraged by the definitions. These concepts were developed in the course of the analysis of quadratic differential equations mentioned earlier [16] to which the reader is referred for a lengthier discussion as well as some of the more technical proofs.

A point and a matrix will refer to elements of $\mathbb{R}^2$, and $\mathbb{R}^{2\times 2}$, respectively, both considered as vector spaces over $\mathbb{R}$. If a and b are points, then $[a\ b]$ will denote the matrix whose first and second columns are given by the coordinates of a and b, respectively.[1] If $\{v_1,\ldots,v_n\} \subseteq V$ are elements of a vector space, then let $\langle v_1 \ldots v_n\rangle$ denote their linear span. A line is the span of a single element, $\langle v_1\rangle$, and a ray, $\langle v_1\rangle^+$, denotes the set $\{\alpha v \mid \alpha\in\mathbb{R}^+\}$. If $\{A_1,\ldots,A_n\}$ is a collection of matrices, then $\{A_1,\ldots,A_n\}x$ denotes the set of points whose i^{th} element is $A_i x$.

Defining the skew symmetric matrix,

$$J \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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[1] We will generally denote scalars by lower case greek letters, points by lower case roman letters, and matrices by upper case roman letters.

we have $x_\perp \triangleq Jx$, the orthogonal complement of x, and $y^T x_\perp = y^T Jx = |x,y|$, where the last symbol denotes the determinant of the matrix [x y]. Let A_s denote the symmetric part of the matrix A. If A is singular then $A = ab^T$ for some a, b $\in \mathbb{R}^2$, and if $A_s = P$, then $x^T Px = 0$ if and only if x is orthogonal to either a or b - i.e. $x \in \langle a_\perp \rangle \cup \langle b_\perp \rangle$. Given a matrix A, the <u>cone generated by A</u>, is the set $\{x \in \mathbb{R}^2 \mid x^T Ax > 0\}$ and is denoted Co(A). The cone of an indefinite symmetric matrix evidently consists of the set of lines lying in between its two <u>zero lines</u> which are defined to be the lines whose points satisfy the equality $x^T Ax = 0$.

For ease of discussion we divide $\mathbb{R}^{2\times 2}$ into three classes based upon spectral properties as follows: a matrix is said to be <u>nodal</u> if it has at least two distinct real eigenvectors; <u>critical</u> if it has an eigenvector of multiplicity two, and a unique eigenspace; or <u>focal</u> if it has no real eigenvectors. Since $|Ax,x| = 0$ if and only if x is an eigenvector of A, that matrix is focal, critical, or nodal if and only if $|Ax,x|$ is sign definite, semi-definite, or indefinite, respectively.[2] Since $|Ax,x| = x^T JAx$, the identical sign conditions on $[JA]_s$ specify whether A is focal, nodal, or critical as well. Notice that e is an eigenvector of A if and only if $[JA]_s = [e_\perp a^T]_s$ for some $a \in \mathbb{R}^2$. Some further properties of matrices are presented in the Appendix , and referred to as "facts" throughout the paper. The trace of a matrix A will be denoted tr{A}. The transposed cofactor matrix defined by A will be denoted $A^\# \triangleq J^T A^T J$.

Given the matrix pair, (A,D), its <u>pencil</u> is the affine line in $\mathbb{R}^{2\times 2}$ given as $A+\mu D$ where $\mu \in \mathbb{R}$. Let I_f, I_c, I_n, denote the subsets of $\mathbb{R}$ on which a

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[2] Of course this determinant vanishes everywhere for the identity matrix.

pencil takes focal, critical, or nodal values, respectively. We have the following characterization of a pencil on I_n and I_c.

Lemma 2.1: Define $\mu(x) \triangleq -\frac{\langle Ax, x\rangle}{\langle Dx, x\rangle}$ and $\lambda(x) \triangleq -\frac{\langle Ax, Dx\rangle}{\langle Dx, x\rangle}$. Then for all points in $\mathbb{R}^2$, x is an eigenvector of $A+\mu(x)D$ with eigenvalue given by $\lambda(x)$ [16].

In keeping with our intent to relate these results to the existing nonlinear controllablity literature, we mention that the Lie Algebra generated by the matrices A,D, denoted $\{A,D\}_{LA}$ (in the notation of [4]), is the smallest linear subspace in $\mathbb{R}^{2\times 2}$ containing A, D, and closed under the lie bracket [1], a matrix product defined by $[X,Y] \triangleq XY-YX$. Recent work [10, 12] indicates that an abstract test for the controllability of certain nonlinear systems may be effected using the Lie Algebra generated by their defining vector fields. These conditions are designed to check when the "drift term" (analogous to the vector field Ax in equation (1)) allows the field to be pointed into either of two half spaces defined by a hypersurface in $\mathbb{R}^n$ containing the reduced degrees of freedom available to direct control. In our problem, this hypersurface is simply the phase curve of the LTI system defined by the matrix D, hence a similar test can be effected with no appeal to Lie Algebras.[3]

Instead, we resort to standard notions from the theory of differential equations. Defining the differential equation

$$\dot{x} = f(x,\tau) \qquad x\epsilon X \tag{2}$$

[3]Note the distinction between our problem and that posed in [12]: we consider all homogeneous bilinear systems in $\mathbb{R}^2$-{0} while the latter considers nonsingular analytic vector fields on a simply connected domain of $\mathbb{R}^n$.

let f be analytic, and denote the integral curve through a point x_0 at the times $I \subseteq \mathbb{R}$ as $\bar{\Phi}_f(I)x_0$. We will say that a subset, $S \subseteq X$ is called a <u>positive invariant set</u> of system (2) if for any $x \in S$, the future trajectory, $\bar{\Phi}_f(\mathbb{R}^+)x$ is contained in S. Clearly, a bilinear system possessed of a set which remains positive invariant for any choice of u is not controllable. To test for positive invariance, it suffices to investigate the effect of the field vector at the boundary of a set. In order to talk about boundary curves and interiors algebraically we will find the following concepts helpful.

Let C be a smooth connected curve in $\mathbb{R}^2$, with forward direction[4] specified by tangent vector $t(x)$, such that $\| t(x) \| = 1$ at every $x \in C$. Then the <u>normal</u> of C at x is denoted $n(x) \triangleq \pm Jt(x)$ and $F(x) \triangleq [t(x)\ n(x)]$ is called the <u>frame</u> of C at x. Specifying the sign of $n(x)$, fixed for all $x \in C$, determines an <u>orientation</u>. When the need to distinguish the tangents, normals, and frames of distinct curves arises, we will write $t_C(x)$, and so on for n and F. If S is an open, proper, simply connected subset of $\mathbb{R}^2$ with smooth boundary, ∂S, then the <u>externally oriented frame of S</u> is the frame (or set of frames if ∂S is not connected), F, with orientation(s) chosen such that for all $x \in \partial S$, $x + \delta n(x) \notin S$ when $\delta > 0$ is arbitrarily small. Given a curve, C, with frame, F, and a vector field, f, we will say that, f has <u>a fixed orientation with respect to C</u> if the inner product $f(x)^T n(x)$ is sign definite or semi-definite for all $x \in C$. The following simple results are stated without proof. Although merely a consequence of the foregoing definitions, they are of central importance in the sequel.

<u>Proposition 2.2:</u> If the vector field in (2) has a fixed

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[4] Formally, C is the image of some real interval, I, under the smooth mapping $x: I \to C$, and the forward direction in C is specified by the ordering of I.

orientation with respect to a smooth curve $C \subseteq \mathbb{R}^2$, then there is no $x\epsilon C$ and $\tau > 0$ such that $\bar{\Phi}_f\{(-\tau,\tau)\}x$ contains points on either side of C.

Corollary 2.2.1: Let S be an open, proper, simply connected, smoothly bounded subset of $\mathbb{R}^2$, with externally oriented frame, $F=[t\ n]$. If f has fixed orientation with respect to ∂S then either S or its complement is positive invariant under system (2)

3. Complete Controllability.

Call a system completely controllable if any point in the state space may be steered to any other point in finite time by an appropriate control action, $u(\tau)$. Since the state space of system (1) , $\mathbb{R}^2-\{0\}$, is not simply connected, it is clear that complete controllability must entail an ability to transfer any ray to any other ray of $\mathbb{R}^2$. If, in addition, radial control - the ability to move toward the origin or away from the origin on a given ray - is available, then it is reasonable to expect that complete controllability holds. This section is devoted to the formal verification of such intuitive reasoning.

Two observations concerning LTI systems in $\mathbb{R}^2$ provide crucial insight into this problem. First, a focal linear system has integral curves which pass through every ray on the plane. Clearly, if the range of the pencil includes a focal value at some $\mu_0\epsilon\mathbb{R}$, then system (1) transfers any ray to any other ray on the plane in finite time for $u \equiv \mu_0$. The importance of focal pencil values will be demonstrated in Proposition 3.1, below. However, as conjectured above, this is not enough: the following system has the accessibility property and its pencil admits focal values, yet it is uncontrollable.

Example 1.

$$\dot{x} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x + u\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}x.$$

Second, a nodal linear system has radially invariant integral curves (on the eigenvector lines) whose forward direction is specified by the sign of its eigenvalues: these may be arbitrarily chosen when the system is controllable. In the present context, while an (A,D) pencil evidently lacks the necessary degrees of freedom to take arbitrary eigenvalues, the ability to assign eigenvalues with both positive and negative real part over a particular interval will be a crucial factor in the controllability behavior of system (1). If $I \subseteq \mathbb{R}$, we will say that a pencil, $A+\mu D$, enjoys the <u>stability assignment property over</u> I, or is <u>assignable over</u> I, if it has both stable and unstable eigenvalues as μ ranges over I.

With this motivation in mind, we may state sufficient conditions for controllability.

<u>Proposition 3.1:</u> If the pencil $A+\mu D$ attains focal values and has the stability assignment property over $\mathbb{R}$ then system (1) is completely controllable.

Proof: Fix any $x_1, x_2 \in \mathbb{R}^2-\{0\}$. Let the pencil be focal at μ_0 so that for $u(\tau) \equiv \mu_0$, the system reaches any ray in $\mathbb{R}^2$ in finite time from any initial condition.

If ω is the natural frequency of $A+\mu_0 D$ then let $\tau \triangleq \omega / 2\pi$, and define the spiral curve $C_i \triangleq \Phi_{h(\mu_0)}([0,\tau])x_i$ or $\Phi_{h(\mu_0)}([-\tau,0])x_i$ depending upon whether x_i is a source or destination point. C_i defines a bounded region, $R_i \subseteq \mathbb{R}^2$ containing the origin and containing (or excluding) the point x_j for one (or the other) index assignment. Let us suppose that $x_1 \in R_2$, hence $x_2 \notin R_1$. With no loss of generality, we

will consider only the case that $A+\mu_0 D$ is unstable, since the alternative case has an identical proof. Under that assumption, and since the pencil has the stability assignment property on $\mathbb{R}$, it either includes saddle values - i.e. with negative determinant - or totally stable values.

In the former case, if $|A+\mu_1 D|<0$ for some $\mu_1 \in \mathbb{R}$, then $\Phi_{h(\mu_1)}\{\mathbb{R}\}x_1$ runs through x_2 and intersects the previously constructed spiral C_2 at two points, a and b, such that $\Phi_{h(\mu_1)}\{\mathbb{R}^+\}a$, the forward solution of system (1) for $u \equiv \mu_1$ starting at a, passes through x_1 and then b (see Figure 1.a). Evidently, choosing u to be one value of μ and then the other, for an appropriate length of time will transfer x_i to x_j, or vice-versa.

In the latter case, $\Phi_{h(\mu_1)}\{\mathbb{R}\}x_j$ is a curve through x_j connecting the origin with the point at infinity, hence must intersect the spiral C_i at a single point, a. To bring x_2 to x_1, travel on C_2 to a, using $u\equiv\mu_0$, and then on $\Phi_{h(\mu_1)}\{\mathbb{R}^+\}a$ to x_1, since $A+\mu_1 D$ is stable and x_1 lies in between a and the origin on that curve. To bring x_1 to x_2, travel away from the origin on $\Phi_{h(\mu_0)}\{\mathbb{R}^+\}x_1$ until leaving the compact set R_2, which must occur at finite time since $A+\mu_0 D$ is unstable. Continue on this trajectory until the reaching a point as $\Phi_{h(\mu_1)}\{\mathbb{R}^-\}x_2 - R_2$. Then apply $u\equiv\mu_1$ to travel from a to x_2 on $\Phi_{h(\mu_1)}\{\mathbb{R}^+\}a$ (see Figure 1.b).

$\square$

In fact, as intuition would suggest, the conditions of Proposition 3.1 are necessary for controllability, and we will devote the rest of this section to the demonstration of that result. To do so, we require a more algebraic characterization of when those conditions fail. For both the assignability

Figure 1a

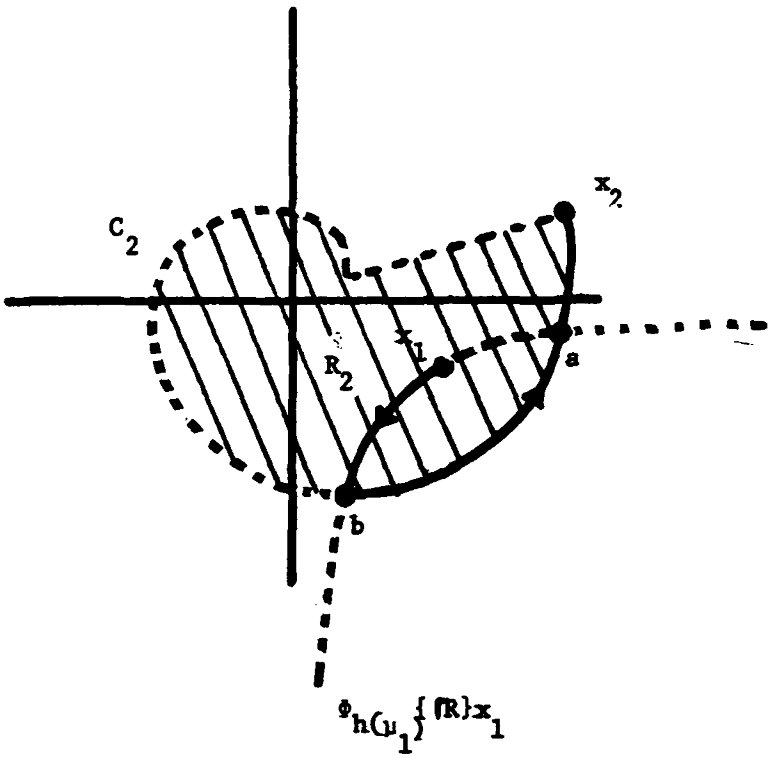
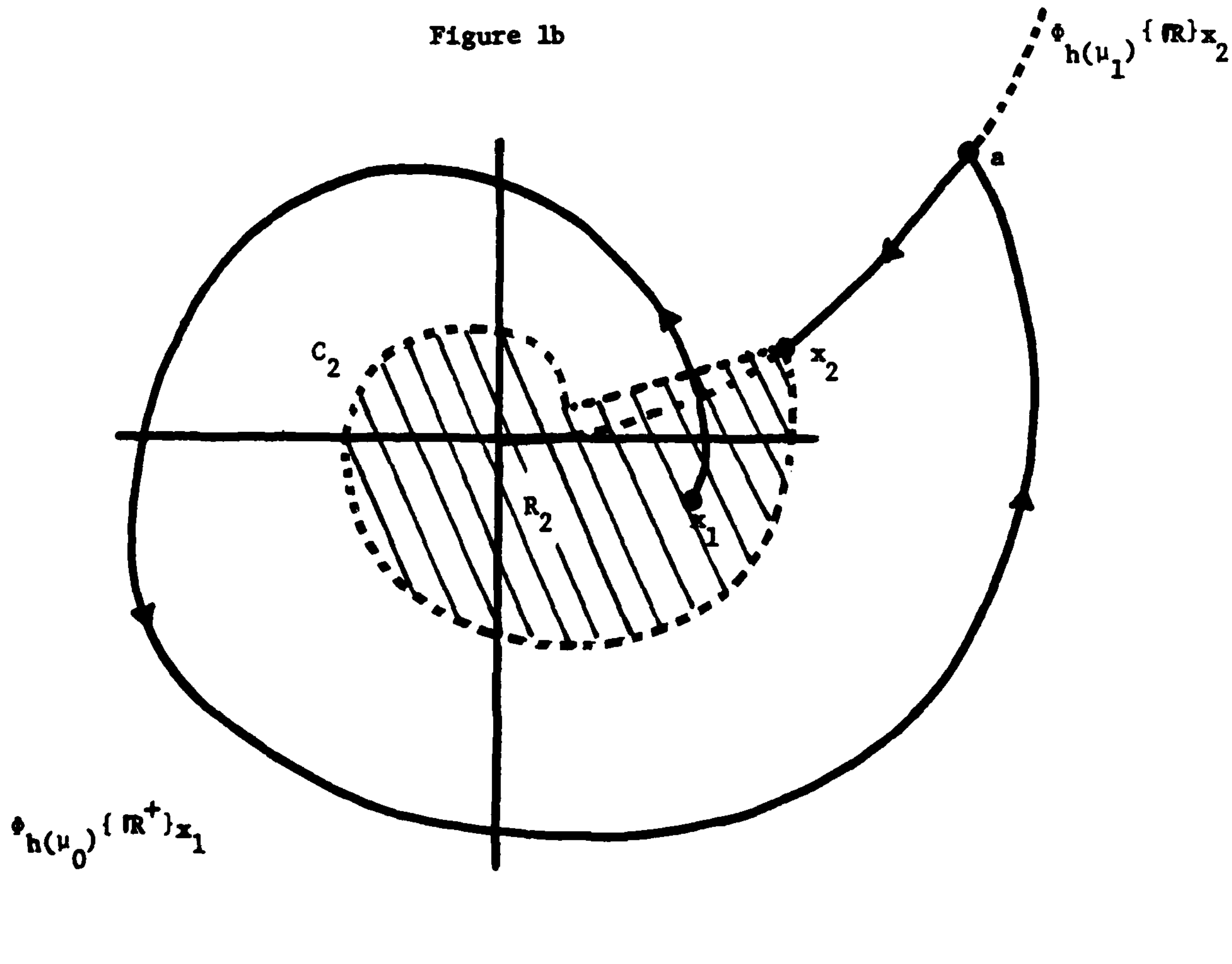


Figure 1b



and focal properties, such a characterization leads almost immediately to the desired results.

Lemma 3.2: The pencil $A + \mu D$ does not enjoy the stability assignment property over $\mathbb{R}$ if and only if D has pure imaginary eigenvalues and $[D^T JA]_s$ is sign definite or semi-definite.

Proof: If D has eigenvalues with non-zero real part then either $tr\{D\} \neq 0$ or $|D| < 0$. In the former case $tr\{A+\mu D\}$ evidently changes sign for $\mu \in \mathbb{R}$, while in the latter $|A+\mu D| = |A| + \mu tr\{D^{\#}A\} + \mu^2 |D|$ (from Fact I.1) evidently takes negative values for large enough μ. Both situations imply assignability, hence that property fails only when D has eigenvalues on the imaginary axis in the complex plane.

In that case, since the pencil is not assignable on $I_f \cup I_c$, the property holds on $\mathbb{R}$ if and only if it holds on I_n. From Lemma 2.1 this is equivalent to the condition that $\lambda(x) \triangleq - \frac{|Ax, Dx|}{|Dx, x|}$ change sign on $\mathbb{R}^2$. When $|Dx,x|$ is sign definite this, in turn is equivalent to the condition that $|Ax,Dx| = x^T D^T JAx$ change sign on $\mathbb{R}^2$ - i.e. that $[D^T JA]_s$ is indefinite. If D has pure imaginary eigenvalues, then $|Dx,x|$ is indeed sign definite unless, for some point d, $D = [dd_{\perp}^T]$, hence $[D^T JA]_s = [d_{\perp} d_{\perp}^T A]_s$ and $\lambda(x) = d_{\perp}^T Ax / d_{\perp}^T x$. This functional changes sign if and only if $d_{\perp}^T A \notin \langle d_{\perp}^T \rangle$ which is equivalent to the condition that $[D^T JA]_s$ be indefinite.

$\square$

Given this understanding of stability assignment, its necessity for complete controllability is quite easy to establish. The reader may note that the following proof is equivalent to a stability argument using the Lyapunov function, v.

<u>Proposition 3.3:</u> If $A + \mu D$ does not have the stability assignment property then (1) is not completely controllable.

Proof: If the pencil is not assignable then according to Lemma 3.2 D has pure imaginary eigenvalues, and $[D^T J]_s = D^T J$ (from Fact I.2) is sign definite (assume positive definite with no loss of generality). Defining $v(x) \triangleq x^T D^T J x$ yields a set $S \triangleq v^{-1}[0,\gamma]$ (γ is any positive constant) whose external frame is given by $F(x) = [Dx \; JDx]/\|Dx\|$, for all $x \in \partial S$. Hence $n(x)^T h(x,u) = -(JDx)^T[A + u(\tau)D]/\|Dx\|$ $= x^T[D^T JA]_s x/\|Dx\|$ which must be definite or semi-definite, again according to the lemma above, and the field has a fixed orientation with respect to ∂S.[5] Thus, by Corollary 2.2.1, S is positive invariant, and the system is not controllable.

$\square$

Similarly, once we characterize the algebraic relations on the pair (A,D) which prevent its pencil from attaining focal values, the necessity of this condition becomes apparent as well. It is interesting to note that the ability to attain focal values is intimately related to the conditions for surjectivity of a quadratic map from $\mathbb{R}^2$ to itself. This may be seen as follows. According to Lemma 2.1 the image of the map $\mu(x) \triangleq -\left|\frac{Ax,x}{Dx,x}\right|$ specifies the subset of $\mathbb{R}$ on which an (A,D) pencil takes real eigenvalues. Clearly, a pencil has no focal values if and only if μ - the ratio of two quadratic forms - is surjective. This ratio maps lines in the plane into the real line, and the conditions under which it is surjective may be given as follows.

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[5] Equivalently, v is a Lyapunov function with definite or semi-definite derivative $\dot{v} = n^T h$

Lemma 3.4: The ratio of two quadratic forms, $\mu(x) = x^T Hx/x^T Gx$ is surjective on $\mathbb{R}$ if and only if both G and H are indefinite and either

(i) their distinct zero lines alternate around the plane: i.e. $x^T Hx = 0$ has a solution both in Co(G) and Co(-G), and vice versa.

or

(ii) they share a zero line in common

Proof: μ is surjective if and only if the pencil (G,H) has no sign definite values - i.e. if and only if $x^T[G + \alpha H]x = 0$ has a non-trivial solution for every $\alpha\epsilon\mathbb{R}$. This is equivalent to the condition that $0 \geq |G+\alpha H| = |G| + \alpha^2|H| + \alpha tr(G^{\#}H)$ according to I.1. The latter implies that $|G|$ and $|H|$ must be negative, and since each is symmetric, this, in turn, implies that G and H are indefinite; that all cones Co($\pm$G) and Co($\pm$H) are non-empty; and that there exist four points, a_1, a_2, d_1, d_2, which define the two zero lines of G and H, respectively (see section 2): $G^{\#} = [a_1 a_2^T]_s$, and $H^{\#} = [d_1 d_2]_s$.

Defining $R \triangleq [d_1\ d_2]$, we have $|R|^2|G+\alpha H| =$

$$|R^T[G+\alpha H]R| \triangleq \left| \begin{bmatrix} d_1^T Gd_1 & d_2^T[G+\alpha H]d_1 \\ d_1^T[G+\alpha H]d_2 & d_2^T Gd_2 \end{bmatrix} \right|$$

$=(d_1^T Gd_1)(d_2^T Gd_2) - (d_2^T[G+\alpha H]d_1)^2$ thus it is sufficient that $(d_1^T Gd_1)(d_2^T Gd_2) \leq 0$ to insure $|G+\alpha H| \leq 0$ for all α. It is also necessary, since $|G+\alpha H| = (d_1^T Gd_1)(d_2^T Gd_2)$ for $\alpha = (d_2^T Gd_1)/(d_2^T Hd_1) = (d_2^T Gd_1)/(d_2^T Jd_2)^2$. Since d_1 and d_2 are the zero lines of H, the condition that $(d_1^T Gd_1)(d_2^T Gd_2) < 0$ implies (i), above, while

$(d_1^T G d_1)(d_2^T G d_2) = 0$ implies (ii).

∎

The reader should note that case (ii) occurs only in a degenerate situation where both numerator and denominator of $\mu(x)$ share a common linear factor, $c^T x$. Defining a quadratic map from $\mathbb{R}^2$ into itself based upon the functional μ,

$$Q(x) \triangleq \begin{bmatrix} x^T G x \\ x^T H x \end{bmatrix},$$

it is clear that in this degenerate case there is some matrix, B, such that $Q(x) = c^T x Bx$. Such a map is never surjective. Using further insights from the previous lemma, we have the following interesting result.

<u>Corollary 3.4.1:</u> Q is surjective on $\mathbb{R}^2$ if and only if the quadratic forms $x^T G x$ and $x^T H x$ have no common factors and μ is surjective on $\mathbb{R}$.

Proof: The condition that $x^T[G+\alpha H]x = 0$ have a non-trivial solution for all $\alpha \in \mathbb{R}$ is equivalent to the condition that $Q(x) \in \langle y \rangle$ for every point $y \triangleq \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$, and this is equivalent to the surjectivity of μ according to the previous lemma. Since Q is homogeneous, i.e. $Q(\lambda x) = \lambda^2 Q(x)$ for any scalar λ, the assumption $Q(x) \in \langle y \rangle$ and $x \neq 0$ implies $\langle y \rangle^+$ is in the image of Q. Hence it suffices to find the conditions under which for every choice of y with $Q(x) \in \langle y \rangle$, there exists another point in the pre-image, of $\langle y \rangle$, z, such that $Q(z) \in \langle -y \rangle^+$.

Clearly if $Q = c^T x Bx$ for some point c and matrix B then Q cannot be surjective, hence it is necessary that G and H have no common linear factors. In the proof of the previous lemma it was shown that under these hypotheses, $G + \alpha H$ is similar to an indefinite symmetric

matrix whose diagonal terms have opposite sign under the change of basis given by the zero lines of H, $R=[d_1\ d_2]$ (which is indeed a basis since $|H| < 0$), regardless of the choice of α. It is easily seen that the two distinct zero lines of such a matrix, $\langle x\rangle$ and $\langle z\rangle$, lie in opposite quadrants, hence the zero lines of $G+\alpha H$ lie in opposite H-cones - i.e. $x \in Co(H)$ implies $z \in Co(-H)$ - and the desired result follows.

$\square$.

Corollary 3.4.2: An (A,D) pencil has no focal values on $\mathbb{R}$ if and only if the quadratic map $Q(x) \triangleq \begin{bmatrix} |Ax,x| \\ |Dx,x| \end{bmatrix}$ is surjective or has the "degenerate structure" $Q(x)=c^T xBx$ *for some point* c and matrix B.

Proof: Defining $G \triangleq [JA]_s$, and $H \triangleq [JD]_s$, $Q(x)$ may be written as $\begin{bmatrix} x^T Gx \\ x^T Hx \end{bmatrix}$. From Lemma 2.1 the pencil has focal values if and only if μ is not surjective, and the result follows from above.

$\square$

The geometric interpretation of this result is that an (A,D) pencil fails to have focal values only when A and D share an eigenvector or their eigenvectors are "interwoven" on the plane. In the former case it is clear that such a line is positive invariant under system (1) for any choice of u. We will show in Proposition 4.1 that this "degenerate" situation is exactly the condition under which the Lie Algebra generated by $\{A,D\}$ does not span $\mathbb{R}^2$ at every point. In the latter case we find that one of the cones defined by the eigenvectors of D, $Co(JD)$, is positive invariant. This may be shown as follows.

Proposition 3.5: If the pencil $A+\mu D$ has no focal values, then

system (1) is not completely controllable on $\mathbb{R}^2-\{0\}$.

Proof: According to the 3.4and its corollaries, if the pencil is never focal then both A and D are nodal, and $|Ax,x|$ has opposite sign (or is zero) on the two distinct eigenvectors of D, d_1 and d_2. The matrix $[JD]_s = [d_1 d_{2\perp}{}^T]_s$ is indefinite, hence Co(JD) has an open interior and possesses a set of four externally oriented frames. Since its boundary, $\partial Co(JD) = \langle d_1\rangle \cup \langle d_2\rangle - \{0\}$, is disconnected, these are given by[6]

$$F(\langle \pm d_i\rangle^+) = \pm\,[d_i \quad (-1)^i J d_i] \text{ for } i=1,2.$$

Since

$$n^T h = (-1)^i d_i{}^T J^T [A+uD] d_i = -(-1)^i |Ad_i, d_i|$$

on $\partial Co(JD)$ it follows that h has a negative definite or semi-definite orientation with respect to either Co(JD) or Co(-JD). System (1) is not controllable.

$\blacksquare$

Taken together, the three propositions of this section provide a proof of the central result.

Theorem 1: System (1) is completely controllable on $\mathbb{R}^2-\{0\}$ if and only if the pencil $A+\mu D$ attains focal values and has the stability assignment property over $\mathbb{R}$.

This result has a computationally more useful formulation as follows.

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[6] We assume, with no loss of generality that both eigenvectors have unit norm, and that both have the same orientation - i.e. $|d_1, d_2| > 0$.

Corollary 3.5.1: System (1) is completely controllable if and only if both

(i) $tr\{D\} \neq 0$ or $|D| < 0$ or $|[D^TJA]_s| < 0$

and

(ii) if $G \stackrel{\Delta}{=} [JA]_s$ and $H \stackrel{\Delta}{=} [JD]_s$, then $|H| \geq 0$ or $|G| \geq 0$ or $|[H^TJG]_s| < 0$

hold

Proof: The first condition follows from Lemma 3.2 directly. The second condition is a restatement of the necessity of the pencil attaining focal values: $A+\mu D$ is focal if and only if $|G+\mu D| = |G| + \mu^2|H| + \mu tr\{H^{\#}G\} > 0$ for some $\mu \in \mathbb{R}$. This is possible if and only if either $|H| \geq 0$ or $|G| \geq 0$ or $tr\{H^{\#}G\}^2 - 4|GH| > 0$ where the last inequality is equivalent to $|[H^TJG]_s| < 0$ according to Fact I.3.

$\square$

4. Related Notions of Controllability

The complexity of nonlinear systems necessitates an attempt to define and study a variety of distinct control behaviors which, in the LTI setting are all subsumed within the question addressed in the foregoing section: the ability to reach the entire state space from any point. In this section we consider a number of such related behaviors whose properties are easily handled by techniques already developed above. We proceed from some weaker notions - definitions of local controllability at a point which may hold even when complete controllability fails - to stronger notions of control ability.

The weakest of these is a property first introduced within the older realizabilty literature mentioned in the introduction, and long known to be necessary for complete controllability. A control system is said to have the

<u>accessibility property</u> (or, to be <u>accessible</u>) if the set of reachable points has a non-empty interior in the state space [23]. It is well known [22] that the accessibility property holds if and only if the Lie Algebra generated by the admissible vector fields (indexed by the range of u) of a control system spans the tangent space at every point of the state space. In our case this is equivalent to the condition that $\langle \{A,D\}_{LA} x \rangle = \mathbb{R}^2$ for all points x. The set of homogeneous bilinear systems in $\mathbb{R}^n$ which satisfy the accessibility condition have been classified by Boothby [3], but we prefer to state and derive the same result for $\mathbb{R}^2$ using language appropriate to this study.

<u>Proposition 4.1:</u> System (1) is accessible if and only if matrices A and D have no eigenvectors in common.

Proof: It suffices to show that $\langle \{A,D\}_{LA} x \rangle = \mathbb{R}^2$ for all points, x, if $A \notin \langle D \rangle$, and A and D have no eigenvector in common. The converse - i.e. if $Ax \in \langle x \rangle$ and $Dx \in \langle x \rangle$, at some point x, or if $A \in \langle D \rangle$, then $\{Ax, Dx\}_{LA}$ does not span $\mathbb{R}^2$ - is trivially true.

Suppose, then, that $Ac = \alpha_1 g$, $Dc = \delta_1 g$, and $R \triangleq [c\ g]$ has full rank. Then

$$\bar{A} \triangleq R^{-1}AR \triangleq \begin{bmatrix} 0 & \alpha_2 \\ \alpha_1 & \alpha_3 \end{bmatrix}$$

$$\bar{D} \triangleq R^{-1}DR \triangleq \begin{bmatrix} 0 & \delta_2 \\ \delta_1 & \delta_3 \end{bmatrix}$$

$$\overline{[A,D]} = [\bar{A},\bar{D}] \triangleq \begin{bmatrix} \alpha_2\delta_1 - \alpha_1\delta_2 & \alpha_2\delta_1 - \alpha_2\delta_3 \\ \alpha_3\delta_1 - \alpha_1\delta_3 & \alpha_2\delta_1 - \alpha_1\delta_2 \end{bmatrix}$$

Since $\bar{c}=\begin{bmatrix}1\\0\end{bmatrix}$, and $\bar{g}=\begin{bmatrix}0\\1\end{bmatrix}$, we have $[A,D]\epsilon\langle g\rangle$ if and only if $\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}=\gamma\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix}$, which we now assume. Note that $[A,D] = \gamma_1 A + \gamma_2 D$ if and only if $\gamma = -\gamma_2/\gamma_1 = \delta_3/\alpha_3$ which is equivalent to the condition that $A\epsilon\langle D\rangle$, hence we further assume that $[A,D]\notin\langle A,D\rangle$.

If four matrices span $\mathbb{R}^{2\times 2}$, i.e. if $\langle A,B,C,D\rangle = \mathbb{R}^{2\times 2}$, then their values at any point, x, must span $\mathbb{R}^2$: $\langle Ax,Bx,Cx,Dx\rangle = \mathbb{R}^2$. Since $\dim\langle A,D,[A,D]\rangle = 3$, it now suffices to show that the next order bracket in the Lie Algebra generation process yields an independent matrix. This follows from inspection: $[D,[A,D]]\epsilon\langle A,D,[A,D]\rangle$ if and only if $\delta_1\delta_2 = 0$ which implies $\alpha_1\alpha_2 = 0$ since $\begin{bmatrix}\delta_1\\\delta_2\end{bmatrix} \epsilon \langle\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}\rangle$, hence, iff A and D share an eigenvector.

0

Given an LTI system, the test for the accessibility property reduces to the familiar rank condition on the controllability matrix. However, in general, as well as for system (1), the ability to reach an open set in the state space from an arbitrary point does not encompass a sufficiently powerful notion of controllability. The reader may note that the conditions for accessibility are exactly those which guarantee against the "degenerate structure" arising in Lemma 3.4 and its corollaries. Simple examples of accessible bilinear systems which are not controllable have been given in the literature [4] and we offer the following.

Example 2.

$$\dot{x} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}x + u\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}x.$$

A more reasonable local condition would be one which insures that any point, posessed of an open reachable set, be contained within the interior of that set. Call such a point <u>locally controllable</u>. We relegate to a later

paper the exhaustive study of this property for system (1) since it leads to an investigation of reachable sets when complete controllability fails[7]. That study is found to be closely allied to a still stronger local concept. From Proposition 3.1 it is apparent that even given a completely controllable system, reaching a nearby point in the state space may first necessitate a complete excursion around the origin in the opposite direction. It is useful to inquire as to when any nearby point may be reached by a trajectory which remains "nearby". This notion has appeared in various guises in the literature: as "local-local controllability" [9]; as "local controllability" [8]; as "small-time-local controllability" [24]. The last concept does not capture the behavior of interest in this setting: since we are permitting arbitrarily large values of the control input, u, for any point x_0, we may reach points arbitrarily close to $\bar{\Phi}_D(\mathbb{R}^+)x_0$ in an arbitrarily small time. That curve may be unbounded, thus it is clear that trajectories permitted by such a definition would not necessarily remain "nearby" x_0 in any sense of the word. The second conflicts with our prior notion of local controllability. Hence we will adopt the first of these and say that a point, x, is <u>local-locally controllable</u> if it is locally controllable and points in any neighborhood may be reached using trajectories remaining in that neighborhood. The conditions for a point to satisfy this requirement are given as follows.

<u>Proposition 4.2:</u> A point, x_0, is local-locally controllable under system (1) if and only if $|Dx_0, x_0| \neq 0$, and for any neighborhood of x_0, N, the pencil of (A,D) is stability assignable on $\mu(N)$ where μ is the functional $\mu(x) \triangleq -\left|\frac{Ax, x}{Dx, x}\right|$.

[7] of course, the property holds trivially at every point when a system is completely controllable

Proof: Since $\mu(N) \subseteq I_n \cup I_c$, the formulae of Lemma 2.1 give the eigenvalues and eigenvectors of the pencil on that inverval: assignability holds on $\mu(N)$ if and only if the eigenvalue functional, $\lambda(x) = -\frac{|Ax, Dx|}{|Dx, x|}$ changes sign on $\mu(N)$. But if $|Dx_0, x_0| \neq 0$ then $\lambda(N)$ changes sign if and only if $|Ax_0, Dx_0| = 0$ and $|Ax, Dx|$ changes sign on every neighborhood of $\langle x_0 \rangle$.

Now assume the hypothesis does not hold. If $|Ax, Dx|$ is definite or semi-definite on small enough neighborhoods of $\langle x_0 \rangle$. Given a fixed $\varepsilon > 0$, the curve $\Phi_D\{(-\varepsilon, \varepsilon)\}x_0$ disconnects any small neighborhood of x_0, N. Since the frame $F(\Phi_D\{(-\varepsilon,\varepsilon)\}x_0) = [Dx\ JDx]/\|Dx\|$, we have $h(x,u)^T n(x) = |Ax, Dx|$ definite or semi-definite for $x \in \Phi_D\{(-\varepsilon,\varepsilon)\}x_0$ and all u. Hence h has a fixed orientation on $\Phi_D\{(-\varepsilon,\varepsilon)\}x_0$, and no trajectory of system (1) can cross that curve inside N according to Proposition 2.2 - x_0 is not local-locally controllable. If $|Ax, Dx|$ changes sign on every neighborhood of x_0, then $|Ax_0, Dx_0| = 0$. In this case, if $|Dx_0, x_0| = 0$, then x_0 is an eigenvector of both A and D, and $\langle x_0 \rangle$ is positive-invariant.

Conversely, assume the hypothesis holds. Choose any neighborhood, N, and for small $\varepsilon > 0$, let $y \triangleq x_0 \pm \varepsilon Jx_0 \in N$. Since $h_y(\mu(y)) = \lambda(y)y$, the state may be steered radially up or radially down on one side or the other of x_0 for rays $\langle y \rangle$ in arbitrarily small neighborhoods of $\langle x_0 \rangle$. Since $|Dx_0, x_0| \neq 0$, $\Phi_D\{(-\delta,\delta)\}x_0$ intersects every such ray. Hence, choosing arbitrarily large values of the control input, the state at time δ, $\Phi_{h(u)}\{\pm\delta\}x_0$, may be brought arbitrarily close to $\Phi_D\{(-\delta,\delta)\}x_0$, and, thereby, to any $\langle y \rangle$ in a neighborhood of $\langle x_0 \rangle$ while remaining N. $\square$

It should be noted that, generically, local-local controllability holds

only for points on the lines containing the eigenvectors of $D^{-1}A$. Clearly, no homogeneous bilinear system (1) is local-locally controllable except on a set with empty interior in $\mathbb{R}^2$.

The final topic we will consider in this section concerns global controllability behavior. If local-local controllability expresses an ability to travel at will without inordinate excursions in the state space, it is reasonable to ask the same question regarding the control space. The problem of control space constraints requires a separate paper of its own, but we will consider here what is perhaps the strongest such constraint that might reasonably be imposed. In Theorem 1 it was shown that any state may be transferred to any other state using no more than two constant control values when the system is completely controllable. We will say that (1) is <u>one-step controllable</u> if it is completely controllable, and any state may be transferred to any other state using a single constant control value.

<u>Proposition 4.3:</u> System (1) is one-step controllable if and only if it has the stability assignment property on I_f.

Proof: Clearly, the system cannot be one-step controllable unless it is completely ontrollable - unless the pencil takes focal values according to Theorem 1. From the results of Proposition 4.2 we may always find an open set of points which are not local-locally controllable. Since system (1) is LTI under constant control inputs, we require that the pencil include values whose associated integral curves return to arbitrarily small neighborhoods of any point. The only such systems are pure centers - i.e. whose matrices have pure imaginary eigenvalues. If $tr(D) \neq 0$ then the pencil cannot have purely imaginary eigenvalues without being assignable on I_f. If $tr(D) = 0$,

then $tr\{A\}$ must vanish as well for the pencil to have a value with pure imaginary roots, hence if it has any such values, then all its values have this property. In this case the pencil is not assignable, hence not controllable.

Conversely, assume that the pencil is assignable on I_f, so that for all $\mu \in I_f$ there exists a continuous regular matrix valued function, $P(\mu)$ such that $A+\mu D = P(\mu)[\alpha(\mu)I+\beta(\mu)J]P^{-1}(\mu)$, and α covers an open neighborhood of the origin in $\mathbb{R}$ with $\alpha(\mu_0) = 0$. It follows that $\exp\{\tau[A+\mu D]\} = e^{\tau\alpha}\,\Omega(\tau,\mu)$ where, for any pair of points, x,y, for every $\mu \in I_f$, there exists a $\tau_0(\mu)$ such that $\tau_n \triangleq \tau_0 + 2\pi n$ implies $\Omega(\tau_n,\mu)x = e^{2\pi n\alpha}\,\gamma(\mu)y$ where γ is some positive continuous function. Setting $r(\mu) \triangleq \ln \gamma/\,2\pi\alpha$ it is clear that r becomes unbounded as μ approaches μ_0, and, hence takes integer values (since it is also continuous) aamong them, say $r(\mu_1) = N$. This implies that $\exp\{\tau_N[A+\mu_1 D]\}x = y$. $\square$

It is interesting to note that assignability over I_f is equivalent to the ability to induce a "Hopf Bifurcation" of the pencil (A,D).

I. Computational Details

<u>Fact I.1:</u> If A and D are two matrices then $|A+D| = |A| + |D| + tr\{A^{\#}D\} = |A| + |D| + tr\{D^{\#}A\}$

<u>Fact I.2:</u> Let D be a matrix with pure imaginary eigenvalues. Then $[DJ]_s = DJ$ is sign definite.

Proof: It will suffice to show that $[DJ]_s = DJ$, since $|DJ| = |D|\,|J| > 0$, under the hypothesis. Since D has pure imaginary eigenvalues it is similar to a multiple of J - i.e. there is a change of basis, P,

such that $PJP^{-1} \varepsilon \langle D \rangle$. Since $P^{-1} = P^{\#}/|P|$, this implies that $PP^TJ \varepsilon \langle D \rangle$, or $DJ \varepsilon \langle PP^T \rangle$. $\square$

<u>Fact I.3:</u> $|P_s| = |P| - tr\{JP\}^2/4$

Proof: $|P+P^T| = 2|P| - tr\{(JP)^2\}$. Since $tr\{(JP)^2\} = -2|P| + tr\{JP\}^2$ (from computation), the result follows.

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